Lecture 18

In this lecture we study mathematical induction, and apply it to the puzzle of the Towers of Hanoi.

Mathematical Induction

Suppose we have a statement

$$P(n)$$

about some integer n. We want to know if $P(n)$ is true for every integer n, from some starting point onwards.

Eg. $$n! > 2^n$$

Is this sometimes true, never true, always true, always true after a certain point, or what?

Check some values :

	$n!$	2^n
$n = 0$:	1	1
$n = 1$:	1	2
$n = 2$:	2	4
$n = 3$:	6	8
$n = 4$:	24	16
$n = 5$:	120	32

At this stage, $n!$ seems to be streaking ahead of 2^n at a great rate.

Inductive reasoning suggests a general rule from studying examples.

Here's a hypothesis:

$$\text{For all } n \geq 4, \quad n! > 2^n.$$

How could we prove this? The idea is to use Proof by Induction. It's based on the following principle.

Principle of Mathematical Induction

$$\left.\begin{array}{l} P(n_0) \\ P(k) \Rightarrow P(k+1) \end{array}\right\} \Rightarrow \begin{array}{l} P(n) \text{ is true} \\ \text{for all } n \geqslant n_0 \end{array}$$

This says that if $P(n)$ is true for some starting value $n = n_0$, and if whenever $P(n)$ is true for some integer k then it's always true for the next integer $k+1$, then $P(n)$ must be true for every integer from n_0 onwards.

So a proof by induction has 3 steps:

<u>Base step</u>: verify that $P(n_0)$ is true.

<u>Inductive step</u>: prove that whenever $P(k)$ is true, $P(k+1)$ is also true.

<u>Conclusion</u>: declare that $P(n)$ is true for every integer $n \geq n_0$.

<u>E.g.</u> Previous example.

Let $P(n)$ be the statement:

$$n! > 2^n .$$

We want to show this is true for all $n \geq 4$.

$\hookrightarrow$ base value n_0

Base step

Verify $P(4)$: $4! > 2^4$

$LHS = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$RHS = 2 \cdot 2 \cdot 2 \cdot 2 = 16$

$\therefore$ $LHS > RHS$ as required

Inductive step

We assume $P(k)$ and try to deduce $P(k+1)$.

$$P(k): \quad k! > 2^k \quad \text{———} \quad (*)$$

(This is called the inductive hypothesis.)

To be proved is $P(k+1)$: $\quad (k+1)! > 2^{k+1}$

$$\begin{aligned} LHS &= (k+1)! \\ &= (k+1)\underbrace{k(k-1)(k-2)\cdots 3\cdot 2\cdot 1}_{k!} \\ &= (k+1)\,k! \end{aligned}$$

$> (k+1)\, 2^k$

$k! > 2^k$ by (*)

$\therefore (k+1)k! > (k+1)\, 2^k$

$> 2 \cdot 2^k$

We started at $n = 4$.

$\therefore k \geq 4$

$\therefore k+1 \geq 5$

$\therefore k+1 > 2$

$\therefore (k+1)\, 2^k > 2 \cdot 2^k$

$= 2^{k+1}$

$=$ RHS

Thus LHS > RHS.

That is, $(k+1)! > 2^{k+1}$ as required.

Conclusion

For all $n \geq 4$, $\quad n! > 2^n$.

The easiest example

We can prove that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for $\underbrace{\text{every positive integer}}_{\text{i.e., for all } n \geq 1}$ n.

starting point

Proof

Base step $(n=1)$

LHS is the sum of n terms.

So when $n=1$, there is only one term:

$$LHS = 1$$

$$RHS = \frac{1(1+1)}{2} = \frac{1\cdot 2}{2} = 1 = LHS$$

Inductive step

Assume $P(k)$: $\quad 1+2+\cdots+k = \dfrac{k(k+1)}{2} \quad$ — $(*)$

Try to prove $P(k+1)$: $\quad 1+2+\cdots+(k+1) = \dfrac{(k+1)(k+2)}{2}$.

$$LHS = 1+2+\cdots+(k+1)$$

$$= \underbrace{1+2+\cdots+k}+(k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \qquad \left[\text{by } (*)\right]$$

$$= (k+1)\left[\frac{k}{2} + 1\right]$$

$$= (k+1)\,\frac{k+2}{2}$$

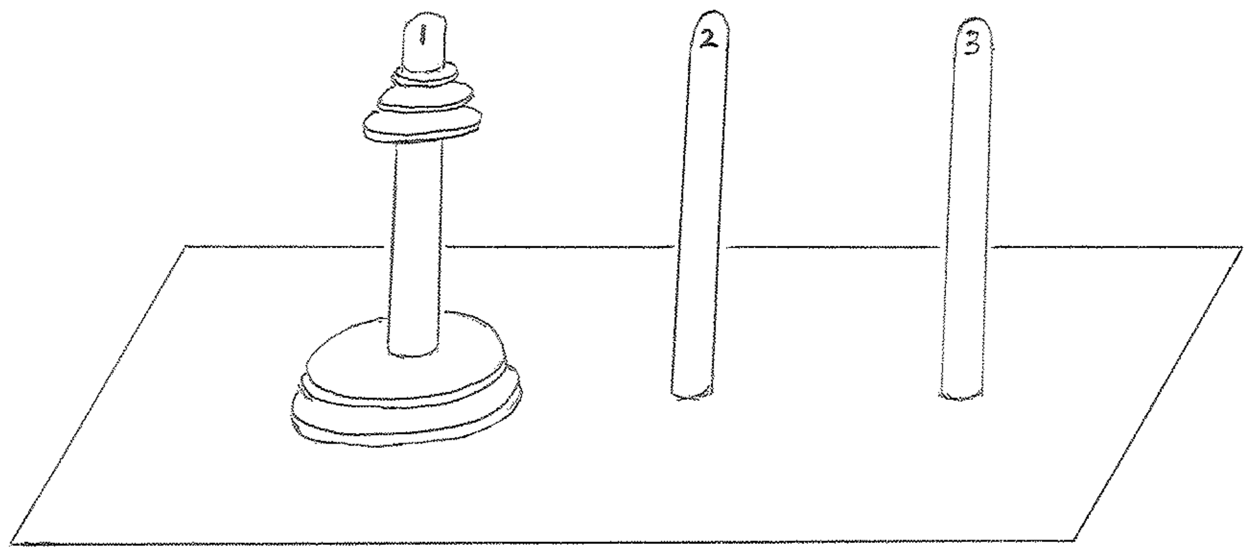
$$= \frac{(k+1)(k+2)}{2}$$

$$= \text{RHS}$$ ✓

<u>Conclusion</u>

For all $n \geq 1$, $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

Example (The Towers of Hanoi)



We need to move all n discs to Pole 3. But we can only move 1 disc at a time, and a disc must never be placed on a smaller disc.

Theorem This can be solved for all $n \geq 1$.

Proof We can use proof by induction.

Base step

Suppose we have 1 disc.

Pick it up and put it on Pole 3. ✓

Inductive step

Suppose we can move $(k-1)$ discs. — $(*)$

We want to show we can move k discs.

Given k discs, leave the big one at the bottom.

Move the other $(k-1)$ to Pole 2 instead of Pole 3. (We know this can be done, by the inductive hypothesis $(*)$.)

Move the big disc to Pole 3. Move the other stack from Pole 2 to Pole 3. (Again, we know this can be done, by $(*)$.)

We've moved the stack of k discs !

Conclusion

For all n, the problem can be solved.